Relaxation Limit in Besov Spaces for Compressible Euler Equations

Jiang Xu[*], Zejun Wang[†]
Department of Mathematics,
Nanjing University of Aeronautics and Astronautics,
Nanjing 211106, P.R.China

Abstract

The relaxation limit in critical Besov spaces for the multidimensional compressible Euler equations is considered. As the first step of this justification, the uniform (global) classical solutions to the Cauchy problem with initial data close to an equilibrium state are constructed in the Chemin-Lerner's spaces with critical regularity. Furthermore, it is shown that the density converges towards the solution to the porous medium equation, as the relaxation time tends to zero. Several important estimates are achieved, including a crucial estimate of commutator.

Keywords: compressible Euler equations, classical solutions, relaxation limit, Chemin-Lerner's spaces

AMS subject classification: 35L25, 35L45, 76N15

1 Introduction and Main Results

In a suitable nondimensional form, the multidimensional compressible Euler equations for a polytropic fluid read as

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0, \\ \partial_t(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) + \nabla P = -\frac{\rho\mathbf{v}}{\tau}. \end{cases} \tag{1.1}$$

Here $\rho = \rho(t,x)$ is the fluid density function of $(t,x)\in[0,+\infty)\times\mathbb{R}^d$ with $d\geq 2$; $\mathbf{v} = (v^1, v^2, \cdots, v^d)^\top$ ($\top$ represents the transpose) denotes the fluid velocity. The pressure $P = P(\rho)$ satisfies the usual γ-law:

$$P(\rho) = A\rho^\gamma (\gamma \geq 1)$$

[*]E-mail: jiangxu_79@yahoo.com.cn
[†]E-mail: wangzejun@gmail.com

where $A > 0$ is some physical constant, the adiabatic exponent $\gamma > 1$ corresponds to the isentropic flow and $\gamma = 1$ corresponds to the isothermal flow; $0 < \tau \le 1$ is a (small) relaxation time. The notation $\nabla, \otimes$ are the gradient operator and the symbol for the tensor products of two vectors, respectively.

In this paper, we are going to study the Cauchy problem of the compressible Euler equations (1.1) subject to the initial data

$$(\rho, \mathbf{v})(0, x) = (\rho_0, \mathbf{v}_0). \tag{1.2}$$

Our first interest is, for fixed $\tau > 0$, to investigate the relaxation effect on the regularity and large-time behavior of classical solutions. As shown by [19, 21], if the initial data are small in some Sobolev space $H^s(\mathbb{R}^d)$ with $s > 1 + d/2$ ($s \in \mathbb{Z}$), the relaxation term which plays the role of damping, can prevent the development of shock waves in finite time and the Cauchy problem (1.1)-(1.2) admits a unique global classical solution. Furthermore, it is proved that the solution in [19] has the L^∞ convergence rate $(1+t)^{-3/2}(d = 3)$ to the constant background state and the optimal $L^p(1 < p \le \infty)$ convergence rate $(1 + t)^{-d/2(1-1/p)}$ in general several dimensions [21], respectively. For the one-dimensional Euler equations with relaxation, the global existence of a smooth solution with small data was proved by Nishida [17], and the asymptotic behavior of the smooth solution was studied in many papers, see e.g. the excellent survey paper by Dafermos [4] and the book by Hsiao [8]. In addition, for the large-time behavior of solutions with vacuum, see [9, 10].

Another main interest is to justify the singular limit as $\tau \to 0$ in (1.1). First, we look at the formal process. To do this, we change the time variable by considering an "$\mathcal{O}(1/\tau)$" time scale:

$$(\rho^\tau, \mathbf{v}^\tau)(s, x) = \Big(\rho, \mathbf{v}\Big)\Big(\frac{s}{\tau}, x\Big). \tag{1.3}$$

Then the new variables satisfy the following equations:

$$\begin{cases} \partial_s \rho^\tau + \nabla \cdot (\frac{\rho^\tau \mathbf{v}^\tau}{\tau}) = 0, \\ \tau^2 \partial_s (\frac{\rho^\tau \mathbf{v}^\tau}{\tau}) + \tau^2 \nabla \cdot (\frac{\rho^\tau \mathbf{v}^\tau \otimes \mathbf{v}^\tau}{\tau^2}) + \frac{\rho^\tau \mathbf{v}^\tau}{\tau} = -\nabla P(\rho^\tau) \end{cases} \tag{1.4}$$

with initial data

$$(\rho^\tau, \mathbf{v}^\tau)(x, 0) = (\rho_0, \mathbf{v}_0). \tag{1.5}$$

At the formal level, if we can show that $\frac{\rho^\tau \mathbf{v}^\tau}{\tau}$ is uniformly bounded, it is not difficult to see that the limit $\mathcal{N}$ of ρ^τ as $\tau \to 0$ satisfies the porous medium equation

$$\begin{cases} \partial_s \mathcal{N} - \Delta P(\mathcal{N}) = 0, \\ \mathcal{N}(x, 0) = \rho_0. \end{cases} \tag{1.6}$$

which is a parabolic equation since $P(\mathcal{N})$ is strictly increasing.

This singular limit problems for hyperbolic relaxation to parabolic equations have attracted much attention. By means of compensated compactness theory, Marcati and his collaborators [13, 14, 16] systematically studied this diffusive limit of generally quasi-linear hyperbolic system, also including the present Euler equations (1.1) for weak solutions. When $\gamma = 1$, Junca and Rascle [11] verified the convergence of solutions to the isothermal equations (1.1) towards the solution to the heat equation for arbitrarily large initial data in $BV(\mathbb{R})$ that are bounded away from the vacuum. Coulombel and Goudon [3] fell back on the classical energy approach and

constructed the uniform smooth solutions to the isothermal Euler equations and justified the relaxation limit in some Sobolev space $H^s(\mathbb{R}^d)(s>1+d/2,\ s\in\mathbb{Z})$ (in x).

In the present paper, we will improve Coulombel and Goudon's work [3] such that the result may hold in the critical space with the regularity index $\sigma=1+d/2$ (a larger space). Indeed, we choose the critical Besov space $B^{\sigma}_{2,1}(\mathbb{R}^d)$ in space-variable x rather than $H^{\sigma}(\mathbb{R}^d)$ as the functional setting, since $B^{\sigma}_{2,1}(\mathbb{R}^d)$ is a subalgebra of $\mathcal{W}^{1,\infty}$. Starting from this simple consideration, based on the Littlewood-Paley decomposition theory and Bony's para-product formula, we first construct the (uniform) global existence of classical solutions in the framework of the Chemin-Lerner's spaces $\widetilde{L}^{\theta}_T(B^s_{p,r})$ in [2], which is a refinement of the usual spaces $L^{\theta}_T(B^s_{p,r})$. Then, using Aubin-Lions compactness lemma, we perform the relaxation limit of (1.1)-(1.2) in Besov spaces.

Main results are stated as follows.

Theorem 1.1. *Let $\bar{\rho}>0$ be a constant reference density. Suppose that $\rho_0-\bar{\rho}$ and $\mathbf{v}_0\in B^{\sigma}_{2,1}(\mathbb{R}^d)(\sigma=1+d/2)$, there exists a positive constant δ_0 independent of τ such that if*

$$\|(\rho_0-\bar{\rho},\mathbf{v}_0)\|_{B^{\sigma}_{2,1}(\mathbb{R}^d)}\leq\delta_0,$$

then the Cauchy problem (1.1)-(1.2) has a unique global solution $(\rho,\mathbf{v})$ satisfying

$$(\rho,\mathbf{v})\in\mathcal{C}^1(\mathbb{R}^+\times\mathbb{R}^d)$$

and

$$(\rho-\bar{\rho},\mathbf{v})\in\widetilde{\mathcal{C}}(B^{\sigma}_{2,1}(\mathbb{R}^d))\cap\widetilde{\mathcal{C}}^1(B^{\sigma-1}_{2,1}(\mathbb{R}^d)).$$

Furthermore, the uniform energy inequality holds

$$\begin{aligned}&\|(\rho-\bar{\rho},\mathbf{v})\|_{\widetilde{L}^{\infty}(B^{\sigma}_{2,1}(\mathbb{R}^d))}\\&+\lambda_0\Big\{\Big\|\frac{1}{\sqrt{\tau}}\mathbf{v}\Big\|_{\widetilde{L}^2(B^{\sigma}_{2,1}(\mathbb{R}^d))}+\Big\|\sqrt{\tau}\nabla\rho\Big\|_{\widetilde{L}^2(B^{\sigma-1}_{2,1}(\mathbb{R}^d))}\Big\}\\\leq\ &C_0\|(\rho_0-\bar{\rho},\mathbf{v}_0)\|_{B^{\sigma}_{2,1}(\mathbb{R}^d)}\end{aligned}\tag{1.7}$$

where $0<\tau\leq1$, λ_0 and C_0 are some uniform positive constants independent of τ.

Remark 1.1. In comparison with that in [3], Theorem 1.1 depends on the low- and high-frequency decomposition methods rather than the classical energy approach. As shown by ourselves [7], the low-frequency estimate of density for the Euler equations (1.1) is absent. Then, we overcame the difficulty by using Gagliardo-Nirenberg-Sobolev inequality (see, e.g., [6]) to obtain a global classical solution, however, the result fails to hold in the critical Besov spaces mentioned above. To obtain the desired result, in the current paper, we add the new context in the proof of global existence. Indeed, some frequency-localization estimates in Chemin-Lerner's spaces are developed, including a crucial estimate of commutator, for details, see Proposition 4.1, Proposition 6.1 and Corollary 6.2.

Based on Theorem 1.1, using the standard weak convergence method and Aubin-Lions compactness lemma in [18], we further obtain the relaxation limit of (1.1)-(1.2) in the larger framework of Besov spaces.

Theorem 1.2. *Let $(\rho, \mathbf{v})$ be the global solution of Theorem 1.1. Then*

$$\rho^\tau - \bar{\rho} \quad \textit{is uniformly bounded in } \ \mathcal{C}(\mathbb{R}^+, B^\sigma_{2,1}(\mathbb{R}^d));$$

$$\frac{\rho^\tau \mathbf{v}^\tau}{\tau} \quad \textit{is uniformly bounded in } \ L^2(\mathbb{R}^+, B^\sigma_{2,1}(\mathbb{R}^d)).$$

Further, there exists some function $\mathcal{N} \in \mathcal{C}(\mathbb{R}^+, \bar{n} + B^\sigma_{2,1}(\mathbb{R}^d))$ which is a global weak solution of (1.6). For any $0 < T, R < \infty$, $\{\rho^\tau(s,x)\}$ strongly converges to $\mathcal{N}(s,x)$ in $\mathcal{C}([0,T], (B^{\sigma-\delta}_{2,1}(B_r))$ as $\tau \to 0$, where $\delta \in (0,1)$ and B_r denotes the ball of radius r in $\mathbb{R}^d$. In addition, it holds that

$$\|(\mathcal{N}(s,\cdot) - \bar{\rho}\|_{B^\sigma_{2,1}(\mathbb{R}^d)} \leq C'_0 \|(\rho_0 - \bar{\rho}, \mathbf{v}_0)\|_{B^\sigma_{2,1}(\mathbb{R}^d)}, \ s \geq 0, \tag{1.8}$$

where $C'_0 > 0$ is a uniform constant independent of τ.

Remark 1.2. Compared with that in [3], the relaxation convergence of classical solutions holds in the Besov spaces with relatively *lower* regularity. To the best of our knowledge, this is the first result for the Euler equations (1.1) in this direction. Therefore, Theorem 1.2 gives a rigorous description that the porous medium equation is usually regarded as an appropriate model for compressible inviscid fluids. In addition, let us also mention that the limit result is generalized to be true for general adiabatic exponent $\gamma \geq 1$ but not the only case $\gamma = 1$ in [3].

The paper is organized as follows. In Section 2, we briefly review the Littlewood-Paley decomposition theory and the characterization of Besov spaces and Chemin-Lerner's spaces. In Section 3, we reformulate the equations (1.1) as a symmetric hyperbolic form in order to obtain the effective frequency-localization estimate and present a local existence result for classical solutions. In Section 4, using the high- and low-frequency decomposition methods, we deduce the frequency-localization estimate in Chemin-Lerner's spaces, which is used to achieve the global existence of uniform classical solutions. Section 5 is devoted to justify the relaxation limit for the Euler equations (1.1). Finally, the paper ends with an appendix, where we give the proof of estimates of commutator.

Notations. Throughout this paper, $C > 0$ is a generic constant independent of τ. Denote by $\mathcal{C}([0,T], X)$ (resp., $\mathcal{C}^1([0,T], X)$) the space of continuous (resp., continuously differentiable) functions on $[0,T]$ with values in a Banach space X. We often label $\|(a,b,c,d)\|_X = \|a\|_X + \|b\|_X + \|c\|_X + \|d\|_X$, where $a,b,c,d \in X$. Here and below, we omit the space dependence for simplicity, since all functional spaces are considered in $\mathbb{R}^d$. Moreover, the integral $\int_{\mathbb{R}^d} f dx$ is labeled as $\int f$ without any ambiguity.

2 Preliminary

For convenience of reader, we try to make the context self-contained, in this section, we briefly review the Littlewood-Paley decomposition theory and some properties of Besov spaces and Chemin-Lerner's spaces. For more details, the reader is referred to [1, 5].

Let (φ, χ) be a couple of smooth functions valued in $[0,1]$ such that φ is supported in the shell $\mathbf{C}(0, \frac{3}{4}, \frac{8}{3}) = \{\xi \in \mathbb{R}^d | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, χ is supported in the ball $\mathbf{B}(0, \frac{4}{3}) = \{\xi \in \mathbb{R}^d | |\xi| \leq \frac{4}{3}\}$ and

$$\chi(\xi) + \sum_{q=0}^{\infty} \varphi(2^{-q}\xi) = 1, \quad \xi \in \mathbb{R}^d.$$

Let $\mathcal{S}'$ be the dual space of the Schwartz class $\mathcal{S}$. For $f \in \mathcal{S}'$, the nonhomogeneous dyadic blocks are defined as follows:

$$\Delta_{-1} f := \chi(D) f = \tilde{\omega} * f \quad \text{with} \ \ \tilde{\omega} = \mathcal{F}^{-1}\chi;$$

$$\Delta_q f := \varphi(2^{-q}D) f = 2^{qd} \int \omega(2^q y) f(x-y) dy \quad \text{with} \ \ \omega = \mathcal{F}^{-1}\varphi, \ \ \text{if} \ \ q \geq 0,$$

where $*$ the convolution operator and $\mathcal{F}^{-1}$ the inverse Fourier transform. The nonhomogeneous Littlewood-Paley decomposition is

$$f = \sum_{q \geq -1} \Delta_q f \qquad \forall f \in \mathcal{S}'.$$

Define the low frequency cut-off by

$$S_q f := \sum_{p \leq q-1} \Delta_p f.$$

Of course, $S_0 f = \Delta_{-1} f$. Moreover, the above Littlewood-Paley decomposition is almost orthogonal in L^2.

Proposition 2.1. *For any $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}'(\mathbb{R}^d)$, the following properties hold:*

$$\Delta_p \Delta_q f \equiv 0 \quad \textit{if} \quad |p-q| \geq 2,$$
$$\Delta_q (S_{p-1} f \Delta_p g) \equiv 0 \quad \textit{if} \quad |p-q| \geq 5.$$

Having defined the linear operators $\Delta_q (q \geq -1)$, we give the definition of Besov spaces and Bony's decomposition.

Definition 2.1. *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. For $1 \leq r < \infty$, Besov spaces $B^s_{p,r} \subset \mathcal{S}'$ are defined by*

$$f \in B^s_{p,r} \Leftrightarrow \|f\|_{B^s_{p,r}} =: \Big(\sum_{q \geq -1} (2^{qs} \|\Delta_q f\|_{L^p})^r \Big)^{\frac{1}{r}} < \infty$$

and $B^s_{p,\infty} \subset \mathcal{S}'$ are defined by

$$f \in B^s_{p,\infty} \Leftrightarrow \|f\|_{B^s_{p,\infty}} =: \sup_{q \geq -1} 2^{qs} \|\Delta_q f\|_{L^p} < \infty.$$

Definition 2.2. *Let f, g be two temperate distributions. The product $f \cdot g$ has the Bony's decomposition:*

$$f \cdot g = T_f g + T_g f + R(f,g),$$

where $T_f g$ is paraproduct of g by f,

$$T_f g = \sum_{p \leq q-2} \Delta_p f \Delta_q g = \sum_q S_{q-1} f \Delta_q v$$

and the remainder $R(f,g)$ is denoted by

$$R(f,g) = \sum_q \Delta_q f \tilde{\Delta}_q g \quad \textit{with} \ \ \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

As regards the remainder of paraproduct, we have the following result.

Proposition 2.2. *Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \le p, p_1, p_2, r, r_1, r_2 \le \infty$. Assume that*

$$\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_2} \le 1, \quad \frac{1}{r} \le \frac{1}{r_1} + \frac{1}{r_2}, \quad \text{and} \quad s_1 + s_2 > 0.$$

Then the remainder R maps $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ in $B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,r}$ and there exists a constant C such that

$$\|R(f,g)\|_{B^{s_1+s_2+d(\frac{1}{p}-\frac{1}{p_1}-\frac{1}{p_2})}_{p,r}} \le \frac{C^{|s_1+s_2|+1}}{s_1+s_2} \|f\|_{B^{s_1}_{p_1,r_1}} \|g\|_{B^{s_2}_{p_2,r_2}}.$$

Some conclusions will be used in subsequent analysis. The first one is the classical Bernstein's inequality.

Lemma 2.1. *Let $k \in \mathbb{N}$ and $0 < R_1 < R_2$. There exists a constant C, depending only on R_1, R_2 and d, such that for all $1 \le a \le b \le \infty$ and $f \in L^a$,*

$$\text{Supp } \mathcal{F}f \subset \mathbf{B}(0, R_1\lambda) \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \le C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a};$$

$$\text{Supp } \mathcal{F}f \subset \mathbf{C}(0, R_1\lambda, R_2\lambda) \Rightarrow C^{-k-1}\lambda^k \|f\|_{L^a} \le \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \le C^{k+1}\lambda^k \|f\|_{L^a}.$$

Here $\mathcal{F}f$ represents the Fourier transform on f.

As a direct corollary of the above inequality, we have

Remark 2.1. For all multi-index α, it holds that

$$\|\partial^\alpha f\|_{B^s_{p,r}} \le C \|f\|_{B^{s+|\alpha|}_{p,r}}.$$

The second one is the embedding properties in Besov spaces.

Lemma 2.2. *Let $s \in \mathbb{R}$ and $1 \le p, r \le \infty$, then*

$$B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,\tilde{r}} \quad \text{whenever} \quad \tilde{s} < s \quad \text{or} \quad \tilde{s} = s \quad \text{and} \quad r \le \tilde{r};$$

$$B^s_{p,r} \hookrightarrow B^{s-d(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r} \quad \text{whenever} \quad \tilde{p} > p;$$

$$B^{d/p}_{p,1}(1 \le p < \infty) \hookrightarrow \mathcal{C}_0, \quad B^0_{\infty,1} \hookrightarrow \mathcal{C} \cap L^\infty,$$

where $\mathcal{C}_0$ is the space of continuous bounded functions which decay at infinity.

The third one is the compactness result for Besov spaces.

Proposition 2.3. *Let $1 \le p, r \le \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. For all $\phi \in C^\infty_c(\mathbb{R}^d)$, the map $f \mapsto \phi f$ is compact from $B^{s+\varepsilon}_{p,r}(\mathbb{R}^d)$ to $B^s_{p,r}(\mathbb{R}^d)$.*

On the other hand, we also present the definition of Chemin-Lerner's spaces first incited by J.-Y. Chemin and N. Lerner [2], which is the refinement of the spaces $L^\theta_T(B^s_{p,r})$.

Definition 2.3. *For $T>0, s\in\mathbb{R}, 1\le r,\theta\le\infty$, set (with the usual convention if $r=\infty$)*

$$\|f\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}:=\Big(\sum_{q\ge -1}(2^{qs}\|\Delta_q f\|_{L^{\theta}_{T}(L^{p})})^{r}\Big)^{\frac{1}{r}}.$$

Then we define the space $\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})$ as the completion of $\mathcal{S}$ over $(0,T)\times\mathbb{R}^{d}$ by the above norm.

Furthermore, we define

$$\widetilde{\mathcal{C}}_{T}(B^{s}_{p,r}):=\widetilde{L}^{\infty}_{T}(B^{s}_{p,r})\cap\mathcal{C}([0,T],B^{s}_{p,r})$$

and

$$\widetilde{\mathcal{C}}^{1}_{T}(B^{s}_{p,r}):=\{f\in\mathcal{C}^{1}([0,T],B^{s}_{p,r})|\partial_t f\in\widetilde{L}^{\infty}_{T}(B^{s}_{p,r})\}.$$

The index T will be omitted when $T=+\infty$. Let us emphasize that

Remark 2.2. According to Minkowski's inequality, it holds that

$$\|f\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}\le\|f\|_{L^{\theta}_{T}(B^{s}_{p,r})}\ \text{if}\ r\ge\theta;\qquad \|f\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}\ge\|f\|_{L^{\theta}_{T}(B^{s}_{p,r})}\ \text{if}\ r\le\theta.$$

Then, we state the property of continuity for product in Chemin-Lerner's spaces $\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})$.

Proposition 2.4. *The following estimate holds:*

$$\|fg\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}\le C(\|f\|_{L^{\theta_1}_{T}(L^{\infty})}\|g\|_{\widetilde{L}^{\theta_2}_{T}(B^{s}_{p,r})}+\|g\|_{L^{\theta_3}_{T}(L^{\infty})}\|f\|_{\widetilde{L}^{\theta_4}_{T}(B^{s}_{p,r})})$$

whenever $s>0, 1\le p\le\infty, 1\le\theta,\theta_1,\theta_2,\theta_3,\theta_4\le\infty$ and

$$\frac{1}{\theta}=\frac{1}{\theta_1}+\frac{1}{\theta_2}=\frac{1}{\theta_3}+\frac{1}{\theta_4}.$$

As a direct corollary, it holds that

$$\|fg\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}\le C\|f\|_{\widetilde{L}^{\theta_1}_{T}(B^{s}_{p,r})}\|g\|_{\widetilde{L}^{\theta_2}_{T}(B^{s}_{p,r})}$$

whenever $s\ge d/p, \frac{1}{\theta}=\frac{1}{\theta_1}+\frac{1}{\theta_2}$.

In addition, the estimate of commutators in $\widetilde{L}^{\theta}_{T}(B^{s}_{p,1})$ spaces is also frequently used in the subsequent analysis. The indices s,p behave just as in the stationary case [1, 5] whereas the time exponent ρ behaves according to Hölder inequality.

Lemma 2.3. *Let $1\le p\le\infty$ and $1\le\theta\le\infty$, then the following inequality is true:*

$$2^{qs}\|[f,\Delta_q]\mathcal{A}g\|_{L^{\theta}_{T}(L^{p})}\le Cc_q\|f\|_{\widetilde{L}^{\theta_1}_{T}(B^{s}_{p,1})}\|g\|_{\widetilde{L}^{\theta_2}_{T}(B^{s}_{p,1})},\quad s=1+d/p,$$

where the commutator $[\cdot,\cdot]$ is defined by $[f,g]=fg-gf$, the operator $\mathcal{A}=\mathrm{div}$ or ∇, C is a generic constant, and c_q denotes a sequence such that $\|(c_q)\|_{l^1}\le 1, \frac{1}{\theta}=\frac{1}{\theta_1}+\frac{1}{\theta_2}$.

Finally, we state a continuity result for compositions to end up this section.

Proposition 2.5. *Let $s>0$, $1\le p,r,\theta\le\infty$, $F\in W^{[s]+1,\infty}_{loc}(I;\mathbb{R})$ with $F(0)=0$, $T\in(0,\infty]$ and $v\in\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})\cap L^{\infty}_{T}(L^{\infty})$. Then*

$$\|F(v)\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}\le C(1+\|v\|_{L^{\infty}_{T}(L^{\infty})})^{[s]+1}\|v\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}.$$

3 Symmetrization and local existence

In terms of the ideas in [19], we introduce a new variable (sound speed) which transforms the equations (1.1) into a symmetric hyperbolic system. For the isentropic case ($\gamma > 1$), denote the sound speed by

$$\psi(\rho) = \sqrt{P'(\rho)},$$

and set $\bar{\psi} = \psi(\bar{\rho})$ corresponding to the sound speed at a background density $\bar{\rho} > 0$. Let

$$\varrho = \frac{2}{\gamma - 1}(\psi(\rho) - \bar{\psi}).$$

Then the Euler equations (1.1) is transformed into the symmetric form for classical solutions:

$$\left\{\begin{array}{l} \partial_t \varrho + \bar{\psi}\mathrm{div}\mathbf{v} = -\mathbf{v}\cdot\nabla\varrho - \frac{\gamma-1}{2}\varrho\mathrm{div}\mathbf{v}, \\ \partial_t \mathbf{v} + \bar{\psi}\nabla\varrho + \frac{1}{\tau}\mathbf{v} = -\mathbf{v}\cdot\nabla\mathbf{v} - \frac{\gamma-1}{2}\varrho\nabla\varrho. \end{array}\right. \tag{3.1}$$

The initial data (1.2) become

$$(\varrho, \mathbf{v})|_{t=0} = (\varrho_0, \mathbf{v}_0) \tag{3.2}$$

with

$$\varrho_0 = \frac{2}{\gamma - 1}(\psi(\rho_0) - \bar{\psi}).$$

Remark 3.1. The variable change is from the open set $\{(\rho, \mathbf{v}) \in (0, +\infty) \times \mathbb{R}^d\}$ to the whole space $\{(\varrho, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^d\}$. It is easy to show that for classical solutions $(\rho, \mathbf{v})$ away from vacuum, (1.1)-(1.2) is equivalent to (3.1)-(3.2) with $\frac{\gamma-1}{2}\varrho + \bar{\psi} > 0$.

Remark 3.2. For the isothermal case $\gamma = 1$, let us introduce the enthalpy change $\varrho(t, x) = \sqrt{A}(\ln\rho - \ln\bar{\rho})$. In this case, the equations (1.1) can be transformed into the system (3.1) with $\gamma = 1$, the reader is referred to [7] for more details. In subsequent sections, we focus mainly on the case $\gamma > 1$, since the isothermal case can be dealt with at a similar manner.

Recently, we have achieved a local existence theory of classical solutions in the framework of Chemin-Lerner's spaces for compressible Euler-Maxwell equations, see [22]. Actually, the new result is applicable to generally symmetrizable hyperbolic systems, including the current Euler equations of special form. Here, we present the result only and the details of the proof are omitted for brevity.

Proposition 3.1. *For any fixed relaxation time $\tau > 0$, assume that $(\varrho_0, \mathbf{v}_0) \in B^{\sigma}_{2,1}$ satisfying $\frac{\gamma-1}{2}\varrho_0 + \bar{\psi} > 0$, then there exists a time $T_0 > 0$ (depending only on the initial data) and a unique solution $(\varrho, \mathbf{v})$ to (3.1)-(3.2) such that $(\varrho, \mathbf{v}) \in \mathcal{C}^1([0, T_0] \times \mathbb{R}^d)$ with $\frac{\gamma-1}{2}\varrho + \bar{\psi} > 0$ for all $t \in [0, T_0]$ and $(\varrho, \mathbf{v}) \in \widetilde{\mathcal{C}}_{T_0}(B^{\sigma}_{2,1}) \cap \widetilde{\mathcal{C}}^1_{T_0}(B^{\sigma-1}_{2,1})$.*

4 Global existence

In this section, we first establish a crucial *a priori* estimate in Chemin-Lerner's spaces. Then by the standard boot-strap argument, we obtain the global existence of classical solutions of (3.1)-(3.2).

The *a priori* estimate is comprised in the following proposition.

Proposition 4.1. *Let $(\varrho,\mathbf{v})\in\widetilde{\mathcal{C}}_T(B^{\sigma}_{2,1})\cap\widetilde{\mathcal{C}}^1_T(B^{\sigma-1}_{2,1})$ be the solution of (3.1)-(3.2) for any given time $T>0$. There exist some positive constants δ_1,λ_1 and C_1 independent of τ such that if*

$$\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^{\sigma}_{2,1})}\leq\delta_1,\tag{4.1}$$

then

$$\begin{aligned}&\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^{\sigma}_{2,1})}\\&+\lambda_1\Big\{\Big\|\frac{1}{\sqrt{\tau}}\mathbf{v}\Big\|_{\widetilde{L}^2_T(B^{\sigma}_{2,1})}+\Big\|\sqrt{\tau}\nabla\varrho\Big\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big\}\\\leq\ &C_1\|(\varrho_0,\mathbf{v}_0)\|_{B^{\sigma}_{2,1}}.\end{aligned}\tag{4.2}$$

Proof. The proof of Proposition 4.1, in fact, is to capture the dissipation rates of $(\varrho,\mathbf{v})$ in turn by using the low- and high-frequency decomposition methods, so we divide it into several steps.

Step 1. The $\widetilde{L}^\infty_T(B^{\sigma}_{2,1})$ estimate of $(\varrho,\mathbf{v})$ and the $\widetilde{L}^2_T(B^{\sigma}_{2,1})$ one of v;

Firstly, we complete the proof of step 1. Applying the localization operator Δ_q to (3.1) yields

$$\begin{cases}\partial_t\Delta_q\varrho+\bar{\psi}\Delta_q\mathrm{div}\mathbf{v}+(\mathbf{v}\cdot\nabla)\Delta_q\varrho=[\mathbf{v},\Delta_q]\cdot\nabla\varrho-\frac{\gamma-1}{2}\Delta_q(\varrho\mathrm{div}\mathbf{v}),\\\partial_t\Delta_q\mathbf{v}+\bar{\psi}\Delta_q\nabla\varrho+(\mathbf{v}\cdot\nabla)\Delta_q\mathbf{v}+\frac{\Delta_q\mathbf{v}}{\tau}=[\mathbf{v},\Delta_q]\cdot\nabla\mathbf{v}-\frac{\gamma-1}{2}\Delta_q(\varrho\nabla\varrho),\end{cases}\tag{4.3}$$

where the commutator$[\cdot,\cdot]$ is defined by $[f,g]=fg-gf$.

Multiplying the first equation of (4.3) by $\Delta_q\varrho$ and the second one by $\Delta_q\mathbf{v}$ respectively, then integrating them over $\mathbb{R}^d$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\Delta_q\varrho\|^2_{L^2}+\|\Delta_q\mathbf{v}\|^2_{L^2}\Big)+\frac{1}{\tau}\|\Delta_q\mathbf{v}\|^2_{L^2}\\=\ &\frac{1}{2}\int\mathrm{div}\mathbf{v}(|\Delta_q\varrho|^2+|\Delta_q\mathbf{v}|^2)+\int\{[\mathbf{v},\Delta_q]\cdot\nabla\varrho\Delta_q\varrho+[\mathbf{v},\Delta_q]\cdot\nabla\mathbf{v}\Delta_q\mathbf{v}\}\\&+\frac{\gamma-1}{2}\int\Delta_q\varrho(\nabla\varrho\cdot\Delta_q v)+\frac{\gamma-1}{2}\int[\varrho,\Delta_q]\nabla\varrho\cdot\Delta_q\mathbf{v}+\frac{\gamma-1}{2}\int[\varrho,\Delta_q]\mathrm{div}\mathbf{v}\Delta_q\varrho.\end{aligned}\tag{4.4}$$

In what follows, we first bound the low frequency part of the quality (4.4). By performing integration by parts and using Hölder- and Gagliardo-Nirenberg-Sobolev inequalities, we have $(d\geq3)$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\Delta_{-1}\varrho\|^2_{L^2}+\|\Delta_{-1}\mathbf{v}\|^2_{L^2}\Big)+\frac{1}{\tau}\|\Delta_{-1}\mathbf{v}\|^2_{L^2}\\\leq\ &\|\mathbf{v}\|_{L^d}\|\Delta_{-1}\varrho\|_{L^{\frac{2d}{d-2}}}\|\Delta_{-1}\nabla\varrho\|_{L^2}+\|\nabla\mathbf{v}\|_{L^\infty}\|\Delta_{-1}\mathbf{v}\|^2_{L^2}\\&+\|[\mathbf{v},\Delta_{-1}]\cdot\nabla\varrho\|_{L^{\frac{2d}{d+2}}}\|\Delta_{-1}\varrho\|_{L^{\frac{2d}{d-2}}}+\|[\mathbf{v},\Delta_{-1}]\cdot\nabla\mathbf{v}\|_{L^2}\|\Delta_{-1}\mathbf{v}\|_{L^2}\\&+\frac{\gamma-1}{2}\|\nabla\varrho\|_{L^d}\|\|\Delta_{-1}\varrho\|_{L^{\frac{2d}{d-2}}}\|\Delta_{-1}\mathbf{v}\|_{L^2}+\frac{\gamma-1}{2}\|[\varrho,\Delta_{-1}]\nabla\varrho\|_{L^2}\|\Delta_{-1}\mathbf{v}\|_{L^2}\\&+\frac{\gamma-1}{2}\|[\varrho,\Delta_{-1}]\mathrm{div}\mathbf{v}\|_{L^{\frac{2d}{d+2}}}\|\Delta_{-1}\varrho\|_{L^{\frac{2d}{d-2}}}\\\leq\ &\|\mathbf{v}\|_{L^d}\|\Delta_{-1}\nabla\varrho\|^2_{L^2}+\|\nabla\mathbf{v}\|_{L^\infty}\|\Delta_{-1}\mathbf{v}\|^2_{L^2}+\|[\mathbf{v},\Delta_{-1}]\cdot\nabla\varrho\|_{L^{\frac{2d}{d+2}}}\|\Delta_{-1}\nabla\varrho\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+\|[\mathbf{v},\Delta_{-1}]\cdot\nabla\mathbf{v}\|_{L^2}\|\Delta_{-1}\mathbf{v}\|_{L^2}+\frac{\gamma-1}{2}\|\nabla\varrho\|_{L^d}\|\|\Delta_{-1}\nabla\varrho\|_{L^2}\|\Delta_{-1}\mathbf{v}\|_{L^2}\\
&+\frac{\gamma-1}{2}\|[\varrho,\Delta_{-1}]\nabla\varrho\|_{L^2}\|\Delta_{-1}\mathbf{v}\|_{L^2}+\frac{\gamma-1}{2}\|[\varrho,\Delta_{-1}]\mathrm{div}\mathbf{v}\|_{L^{\frac{2d}{d+2}}}\|\Delta_{-1}\nabla\varrho\|_{L^2}. \qquad (4.5)
\end{aligned}
$$

Integrating (4.5) with respect to $t\in[0,T]$ implies

$$
\begin{aligned}
&\frac{1}{2}\Big(\|\Delta_{-1}\varrho\|_{L^2}^2+\|\Delta_{-1}\mathbf{v}\|_{L^2}^2\Big)\Big|_0^t+\frac{1}{\tau}\|\Delta_{-1}\mathbf{v}\|_{L^2_t(L^2)}^2\\
\leq\ &\|\mathbf{v}\|_{L^2_t(L^d)}\|\Delta_{-1}\nabla\varrho\|_{L^2_t(L^2)}\|\Delta_{-1}\nabla\varrho\|_{L^\infty_t(L^2)}+\|\nabla\mathbf{v}\|_{L^\infty_t(L^\infty)}\|\Delta_{-1}\mathbf{v}\|_{L^2_t(L^2)}^2\\
&+\|[\mathbf{v},\Delta_{-1}]\cdot\nabla\varrho\|_{L^2_t(L^{\frac{2d}{d+2}})}\|\Delta_{-1}\nabla\varrho\|_{L^2_t(L^2)}+\|[\mathbf{v},\Delta_{-1}]\cdot\nabla\mathbf{v}\|_{L^2_t(L^2)}\|\Delta_{-1}\mathbf{v}\|_{L^2_t(L^2)}\\
&+\frac{\gamma-1}{2}\|\nabla\varrho\|_{L^\infty_t(L^d)}\|\Delta_{-1}\nabla\varrho\|_{L^2_t(L^2)}\|\Delta_{-1}\mathbf{v}\|_{L^2_t(L^2)}\\
&+\frac{\gamma-1}{2}\|[\varrho,\Delta_{-1}]\nabla\varrho\|_{L^2_t(L^2)}\|\Delta_{-1}\mathbf{v}\|_{L^2_t(L^2)}\\
&+\frac{\gamma-1}{2}\|[\varrho,\Delta_{-1}]\mathrm{div}\mathbf{v}\|_{L^2_t(L^{\frac{2d}{d+2}})}\|\Delta_{-1}\nabla\varrho\|_{L^2_t(L^2)}. \qquad (4.6)
\end{aligned}
$$

Then multiplying the factor $2^{-2\sigma}$ on both sides of (4.6), we can get

$$
\begin{aligned}
&\frac{1}{2}2^{-2\sigma}\Big(\|\Delta_{-1}\varrho\|_{L^2}^2+\|\Delta_{-1}\mathbf{v}\|_{L^2}^2\Big)+\frac{2^{-2\sigma}}{\tau}\|\Delta_{-1}\mathbf{v}\|_{L^2_t(L^2)}^2\\
\leq\ &\frac{2^{-2\sigma}}{2}\Big(\|\Delta_{-1}\varrho_0\|_{L^2}^2+\|\Delta_{-1}\mathbf{v}_0\|_{L^2}^2\Big)+Cc_{-1}^2\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\|\nabla\varrho\|_{\widetilde{L}^\infty_T(B^{\sigma-1}_{2,1})}\\
&+Cc_{-1}^2\|\nabla\mathbf{v}\|_{\widetilde{L}^\infty_T(B^{\sigma-1}_{2,1})}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}^2+Cc_{-1}^2\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}^2\\
&+Cc_{-1}^2\|\nabla\mathbf{v}\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\\
&+Cc_{-1}^2\|\nabla\varrho\|_{\widetilde{L}^\infty_T(B^{\sigma-1}_{2,1})}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\\
&+Cc_{-1}^2\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\\
&+Cc_{-1}^2\|\nabla\varrho\|_{\widetilde{L}^\infty_T(B^{\sigma-1}_{2,1})}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}, \qquad (4.7)
\end{aligned}
$$

where we used Remark 2.2, Lemma 2.3 and Corollary 6.2 which will be shown in the Appendix. Here and below $C>0$ denotes a uniform constant independent of τ; $\{c_{-1}\}$ denotes some sequence which satisfies $\|(c_{-1})\|_{l^1}\leq 1$ although each $\{c_{-1}\}$ is possibly different in (4.7).

Next, we turn to estimate the high-frequency part ($q\geq 0$) of the quality (4.4). With the aid of Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\Delta_q\varrho\|_{L^2}^2+\|\Delta_q\mathbf{v}\|_{L^2}^2\Big)+\frac{1}{\tau}\|\Delta_q\mathbf{v}\|_{L^2}^2\\
\leq\ &\frac{1}{2}\|\nabla\mathbf{v}\|_{L^\infty}(\|\Delta_q\varrho\|_{L^2}^2+\|\Delta_q\mathbf{v}\|_{L^2}^2)+\|[\mathbf{v},\Delta_q]\cdot\nabla\varrho\|_{L^2}\|\Delta_q\varrho\|_{L^2}\\
&+\|[\mathbf{v},\Delta_q]\cdot\nabla\mathbf{v}\|_{L^2}\|\Delta_q\mathbf{v}\|_{L^2}+\frac{\gamma-1}{2}\|\nabla\varrho\|_{L^\infty}\|\Delta_q\varrho\|_{L^2}\|\Delta_q\mathbf{v}\|_{L^2}\\
&+\frac{\gamma-1}{2}\|[\varrho,\Delta_q]\nabla\varrho\|_{L^2}\|\Delta_q\mathbf{v}\|_{L^2}+\frac{\gamma-1}{2}\|[\varrho,\Delta_q]\mathrm{div}\mathbf{v}\|_{L^2}\|\Delta_q\varrho\|_{L^2}. \qquad (4.8)
\end{aligned}
$$

By integrating (4.8) with respect to $t \in [0, T]$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\Big(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q \mathbf{v}\|_{L^2}^2\Big)\Big|_0^t + \frac{1}{\tau}\|\Delta_q \mathbf{v}\|_{L_t^2(L^2)}^2 \\
\leq\ & \frac{1}{2}\|\nabla \mathbf{v}\|_{L_t^2(L^\infty)}(\|\Delta_q\varrho\|_{L_t^2(L^2)}\|\Delta_q\varrho\|_{L_t^\infty(L^2)} + \|\Delta_q\mathbf{v}\|_{L_t^2(L^2)}\|\Delta_q\mathbf{v}\|_{L_t^\infty(L^2)}) \\
&+\|[\mathbf{v}, \Delta_q]\cdot\nabla\varrho\|_{L_t^2(L^2)}\|\Delta_q\varrho\|_{L_t^2(L^2)} + \|[\mathbf{v}, \Delta_q]\cdot\nabla\mathbf{v}\|_{L_t^2(L^2)}\|\Delta_q\mathbf{v}\|_{L_t^2(L^2)} \\
&+\frac{\gamma-1}{2}\|\nabla\varrho\|_{L_t^\infty(L^\infty)}\|\Delta_q\varrho\|_{L_t^2(L^2)}\|\Delta_q\mathbf{v}\|_{L_t^2(L^2)} \\
&+\frac{\gamma-1}{2}\|[\varrho, \Delta_q]\nabla\varrho\|_{L_t^2(L^2)}\|\Delta_q\mathbf{v}\|_{L_t^2(L^2)} \\
&+\frac{\gamma-1}{2}\|[\varrho, \Delta_q]\mathrm{div}\mathbf{v}\|_{L_t^2(L^2)}\|\Delta_q\varrho\|_{L_t^2(L^2)}. \qquad (4.9)
\end{aligned}
$$

Then multiplying the factor $2^{2q\sigma}$ on both sides of (4.9) and using Lemma 2.3, we obtain

$$
\begin{aligned}
&\frac{1}{2}2^{2q\sigma}\Big(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q\mathbf{v}\|_{L^2}^2\Big) + \frac{2^{2q\sigma}}{\tau}\|\Delta_q\mathbf{v}\|_{L_t^2(L^2)}^2 \\
\leq\ & \frac{1}{2}2^{2q\sigma}\Big(\|\Delta_q\varrho_0\|_{L^2}^2 + \|\Delta_q\mathbf{v}_0\|_{L^2}^2\Big) + Cc_q^2\|\nabla\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}(\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} \\
&+\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\|\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}) + Cc_q^2\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})} \\
&+Cc_q^2\|\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}^2 + Cc_q^2\|\nabla\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma-1})}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} \\
&+Cc_q^2\|\nabla\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma-1})}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} \\
&+Cc_q^2\|\nabla\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma-1})}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}, \qquad (4.10)
\end{aligned}
$$

where we have used the fact $\|\Delta_q\nabla f\|_{L^2} \approx 2^q\|\Delta_q f\|_{L^2} (q \geq 0)$ derived by Lemma 2.1. The constant $C > 0$ is a uniform constant independent of τ; $\{c_q\}$ denotes some sequence which satisfies $\|(c_q)\|_{l^1} \leq 1$ although each $\{c_q\}$ is possibly different in (4.10).

To conclude, combining (4.7) with (4.10) gives

$$
\begin{aligned}
&\frac{1}{2}2^{2q\sigma}\Big(\|\Delta_q\varrho\|_{L^2}^2 + \|\Delta_q\mathbf{v}\|_{L^2}^2\Big) + \frac{2^{2q\sigma}}{\tau}\|\Delta_q\mathbf{v}\|_{L_t^2(L^2)}^2 \\
\leq\ & \frac{1}{2}2^{2q\sigma}\Big(\|\Delta_q\varrho_0\|_{L^2}^2 + \|\Delta_q\mathbf{v}_0\|_{L^2}^2\Big) + Cc_q^2\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}(\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)} \\
&+\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\|\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}) + Cc_q^2\|\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}^2 \\
&+Cc_q^2\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\ (q \geq -1). \qquad (4.11)
\end{aligned}
$$

By Young's inequality, we have

$$
\begin{aligned}
&2^{q\sigma}\Big(\|\Delta_q\varrho\|_{L_T^\infty(L^2)} + \|\Delta_q\mathbf{v}\|_{L_T^\infty(L^2)}\Big) + \frac{\mu_1 2^{q\sigma}}{\sqrt{\tau}}\|\Delta_q\mathbf{v}\|_{L_T^2(L^2)} \\
\leq\ & C2^{q\sigma}\Big(\|\Delta_q\varrho_0\|_{L^2} + \|\Delta_q\mathbf{v}_0\|_{L^2}\Big) + Cc_q\sqrt{\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)} + \sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\Big) \\
&+Cc_q\sqrt{\|\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^\sigma)}}\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^\sigma)}\ (q \geq -1), \qquad (4.12)
\end{aligned}
$$

where μ_1 is a positive constant independent of τ.

Summing up (4.12) on $q \geq -1$, we immediately get

$$\begin{aligned}
&\|(\varrho, \mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})} + \frac{\mu_1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} \\
\leq\ & C\|(\varrho_0, \mathbf{v}_0)\|_{B^\sigma_{2,1}} + C\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})} + \sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big) \\
&+C\sqrt{\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}. \qquad (4.13)
\end{aligned}$$

Step 2. The $\widetilde{L}^2_T(B^{\sigma-1}_{2,1})$ estimate of $\nabla\varrho$.

Using the second equation of (3.1), we have

$$\bar{\psi}\nabla\varrho = -\Big(\partial_t\mathbf{v} + \frac{1}{\tau}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \frac{\gamma-1}{2}\varrho\nabla\varrho\Big). \qquad (4.14)$$

Apply the operator Δ_q to (4.14) to get

$$\begin{aligned}
&\bar{\psi}\tau\Delta_q\nabla\varrho \\
=\ & -\Big(\tau\Delta_q\partial_t\mathbf{v} + \Delta_q\mathbf{v} - \tau[\mathbf{v}, \Delta_q]\nabla\mathbf{v} + \tau\mathbf{v}\cdot\Delta_q\nabla\mathbf{v} \\
&-\frac{\gamma-1}{2}\tau[\varrho, \Delta_q]\nabla\varrho + \frac{\gamma-1}{2}\tau\varrho\Delta_q\nabla\varrho\Big). \qquad (4.15)
\end{aligned}$$

Integrating the resulting equality over $\mathbb{R}^d$ after multiplying $\Delta_q\nabla\varrho$, we have

$$\begin{aligned}
&\bar{\psi}\tau\|\Delta_q\nabla\varrho\|^2_{L^2} \\
=\ & -\int\Big(\tau\Delta_q\partial_t\mathbf{v} + \Delta_q\mathbf{v} - \tau[\mathbf{v}, \Delta_q]\nabla\mathbf{v} + \tau\mathbf{v}\cdot\Delta_q\nabla\mathbf{v} \\
&-\frac{\gamma-1}{2}\tau[\varrho, \Delta_q]\nabla\varrho + \frac{\gamma-1}{2}\tau\varrho\Delta_q\nabla\varrho\Big)\cdot\Delta_q\nabla\varrho, \qquad (4.16)
\end{aligned}$$

where the first integral can be estimated as

$$\begin{aligned}
-\tau\int\Delta_q\partial_t\mathbf{v}\cdot\Delta_q\nabla\varrho \ =\ & \tau\int\Delta_q\mathrm{div}\partial_t\mathbf{v}\Delta_q\varrho \\
=\ & \tau\frac{d}{dt}\int\Delta_q\mathrm{div}\mathbf{v}\Delta_q\varrho - \tau\int\Delta_q\mathrm{div}\mathbf{v}\Delta_q\partial_t\varrho \\
=\ & \tau\frac{d}{dt}\int\Delta_q\mathrm{div}\mathbf{v}\Delta_q\varrho \\
&-\tau\int\Delta_q\mathrm{div}\mathbf{v}\Delta_q\Big(-\bar{\psi}\mathrm{div}\mathbf{v} - \mathbf{v}\cdot\nabla\varrho - \frac{\gamma-1}{2}\varrho\mathrm{div}\mathbf{v}\Big) \\
\leq\ & \tau\frac{d}{dt}\int\Delta_q\mathrm{div}\mathbf{v}\Delta_q\varrho + \tau\bar{\psi}\|\mathrm{div}\mathbf{v}\|^2_{L^2} + \tau\|\mathbf{v}\|_{L^\infty}\|\Delta_q\mathrm{div}\mathbf{v}\|_{L^2}\|\Delta_q\nabla\varrho\|_{L^2} \\
&+\tau\|\Delta_q\mathrm{div}\mathbf{v}\|_{L^2}\|[\mathbf{v}, \Delta_q]\nabla\varrho\|_{L^2} + \frac{\gamma-1}{2}\tau\|\varrho\|_{L^\infty}\|\mathrm{div}\mathbf{v}\|^2_{L^2} \\
&+\frac{\gamma-1}{2}\tau\|\mathrm{div}\mathbf{v}\|_{L^2}\|[\varrho, \Delta_q]\mathrm{div}\mathbf{v}\|_{L^2}. \qquad (4.17)
\end{aligned}$$

Remark 4.1. In the inequality (4.17), the information behind the mass and momentum equations of (3.1) help us eventually to estimate the term $\partial_t\mathbf{v}$ well. Otherwise, as in [7], we have to establish an auxiliary inequality with respect to the variable $(\varrho_t, \mathbf{v}_t)$ to close the *a priori* estimate, which leads to the tedious proof of global existence consequently.

Together with (4.16)-(4.17), we are led to the estimate

$$\begin{aligned}
&\bar{\psi}\tau\|\Delta_q\nabla\varrho\|_{L^2}^2\\
\leq\ &\tau\frac{d}{dt}\int\Delta_q\mathrm{div}\mathbf{v}\Delta_q\varrho+\tau\bar{\psi}\|\mathrm{div}\mathbf{v}\|_{L^2}^2+\|\Delta_q\mathbf{v}\|_{L^2}\|\Delta_q\nabla\varrho\|_{L^2}\\
&+\tau\|\mathbf{v}\|_{L^\infty}\|\Delta_q\mathrm{div}\mathbf{v}\|_{L^2}\|\Delta_q\nabla\varrho\|_{L^2}\\
&+\tau\|\Delta_q\mathrm{div}\mathbf{v}\|_{L^2}\|[\mathbf{v},\Delta_q]\nabla\varrho\|_{L^2}+\frac{\gamma-1}{2}\tau\|\varrho\|_{L^\infty}\|\Delta_q\mathrm{div}\mathbf{v}\|_{L^2}^2\\
&+\frac{\gamma-1}{2}\tau\|\Delta_q\mathrm{div}\mathbf{v}\|_{L^2}\|[\varrho,\Delta_q]\mathrm{div}\mathbf{v}\|_{L^2}+\tau\|\mathbf{v}\|_{L^\infty}\|\Delta_q\nabla\mathbf{v}\|_{L^2}\|\Delta_q\nabla\varrho\|_{L^2}\\
&+\tau\|[\mathbf{v},\Delta_q]\nabla\mathbf{v}\|_{L^2}\|\Delta_q\nabla\varrho\|_{L^2}+\frac{\gamma-1}{2}\|\varrho\|_{L^\infty}\|\Delta_q\nabla\varrho\|_{L^2}^2\\
&+\frac{\gamma-1}{2}\tau\|[\varrho,\Delta_q]\nabla\varrho\|_{L^2}\|\Delta_q\nabla\varrho\|_{L^2}.
\end{aligned}\tag{4.18}$$

Integrating (4.18) in $t\in[0,T]$ gives

$$\begin{aligned}
&\bar{\psi}\tau\|\Delta_q\nabla\varrho\|_{L_t^2(L^2)}^2\\
\leq\ &\tau\Big(\|\Delta_q\mathrm{div}\mathbf{v}\|_{L^2}\|\Delta_q\varrho\|_{L^2}+\|\Delta_q\mathrm{div}\mathbf{v}_0\|_{L^2}\|\Delta_q\varrho_0\|_{L^2}\Big)+\tau\bar{\psi}\|\Delta_q\mathrm{div}\mathbf{v}\|_{L_T^2(L^2)}^2\\
&+\|\Delta_q\mathbf{v}\|_{L_T^2(L^2)}\|\Delta_q\nabla\varrho\|_{L_T^2(L^2)}+\tau\|\mathbf{v}\|_{L_T^\infty(L^\infty)}\|\Delta_q\mathrm{div}\mathbf{v}\|_{L_T^2(L^2)}\|\Delta_q\nabla\varrho\|_{L_T^2(L^2)}\\
&+\tau\|\Delta_q\mathrm{div}\mathbf{v}\|_{L_T^2(L^2)}\|[\mathbf{v},\Delta_q]\nabla\varrho\|_{L_T^2(L^2)}+\frac{\gamma-1}{2}\tau\|\varrho\|_{L_T^\infty(L^\infty)}\|\Delta_q\mathrm{div}\mathbf{v}\|_{L_T^2(L^2)}^2\\
&+\frac{\gamma-1}{2}\tau\|\Delta_q\mathrm{div}\mathbf{v}\|_{L_T^2(L^2)}\|[\varrho,\Delta_q]\mathrm{div}\mathbf{v}\|_{L_T^2(L^2)}+\tau\|\mathbf{v}\|_{L_T^\infty(L^\infty)}\|\Delta_q\nabla\mathbf{v}\|_{L_T^2(L^2)}\|\Delta_q\nabla\varrho\|_{L_T^2(L^2)}\\
&+\tau\|[\mathbf{v},\Delta_q]\nabla\mathbf{v}\|_{L_T^2(L^2)}\|\Delta_q\nabla\varrho\|_{L_T^2(L^2)}+\frac{\gamma-1}{2}\|\varrho\|_{L_T^\infty(L^\infty)}\|\Delta_q\nabla\varrho\|_{L_T^2(L^2)}^2\\
&+\frac{\gamma-1}{2}\tau\|[\varrho,\Delta_q]\nabla\varrho\|_{L_T^2(L^2)}\|\Delta_q\nabla\varrho\|_{L_T^2(L^2)}.
\end{aligned}\tag{4.19}$$

Multiply the factor $2^{2q(\sigma-1)}$ on both sides of (4.19) to get

$$\begin{aligned}
&\tau 2^{2q(\sigma-1)}\|\Delta_q\nabla\varrho\|_{L_t^2(L^2)}^2\\
\leq\ &C\tau c_q^2\Big(\|\mathrm{div}\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma-1})}\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma-1})}+\|\mathrm{div}\mathbf{v}_0\|_{B_{2,1}^{\sigma-1}}\|\varrho_0\|_{B_{2,1}^{\sigma-1}}\Big)\\
&+C\tau c_q^2\|\mathrm{div}\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}^2+Cc_q^2\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\\
&+C\tau c_q^2\|\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma-1})}\|\mathrm{div}\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\\
&+C\tau c_q^2\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma})}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}^2+C\tau c_q^2\|\mathbf{v}\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma})}\|\mathbf{v}\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma})}\|\nabla m\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}\\
&+C\tau c_q^2\|\varrho\|_{\widetilde{L}_T^\infty(B_{2,1}^{\sigma})}\|\nabla\varrho\|_{\widetilde{L}_T^2(B_{2,1}^{\sigma-1})}^2,
\end{aligned}\tag{4.20}$$

where we used Lemma 2.3, and $\{c_q\}$ denotes some sequence which satisfies $\|(c_q)\|_{l^1}\leq 1$.

Then it follows from Young's inequality that

$$
\begin{aligned}
&\sqrt{\tau}2^{q(\sigma-1)}\|\Delta_q\nabla\varrho\|_{L^2_T(L^2)}\\
\leq\;& Cc_q\Big(\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\Big)+\frac{Cc_q}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\\
&+Cc_q\sqrt{\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&+Cc_q\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\\
&+Cc_q\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})},
\end{aligned}\tag{4.21}
$$

where we have used the smallness of $\tau(0<\tau\leq 1)$.

Finally, summing up (4.21) on $q\geq -1$, we deduce that

$$
\begin{aligned}
&\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\\
\leq\;& C\Big(\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\Big)+\frac{C}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}\\
&+C\sqrt{\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&+C\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+C\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}.
\end{aligned}\tag{4.22}
$$

Step 3. Combining the above analysis.

Combining with (4.13) and (4.22), we end up with

$$
\begin{aligned}
&\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\frac{\mu_1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}K\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\\
\leq\;& C\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}+C\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&+C\sqrt{\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+CK\Big(\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\Big)\\
&+\frac{CK}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+CK\sqrt{\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&+CK\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+CK\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})},
\end{aligned}\tag{4.23}
$$

where $K>0$ is a uniform constant independent of τ. In order to eliminate the term $\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}$ and the singular one $\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}/\sqrt{\tau}$, we take the constant K such that

$$
0<K\leq\min\Big\{\frac{1}{2C},\frac{\mu_1}{2C}\Big\}.
$$

Furthermore, it is not difficult to obtain

$$
\frac{1}{2}\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}+\frac{\mu_1}{2\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}K\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}
$$

$$
\begin{aligned}
&\leq C\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}+C\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&\quad +C\sqrt{\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+CK\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\\
&\quad +CK\sqrt{\|\mathbf{v}\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&\quad +CK\sqrt{\|\varrho\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&\leq C\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}+C\sqrt{\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_T(B^\sigma_{2,1})}}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big)\\
&\leq C\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}+C\sqrt{\delta_1}\Big(\frac{1}{\sqrt{\tau}}\|\mathbf{v}\|_{\widetilde{L}^2_T(B^\sigma_{2,1})}+\sqrt{\tau}\|\nabla\varrho\|_{\widetilde{L}^2_T(B^{\sigma-1}_{2,1})}\Big),
\end{aligned}
\tag{4.24}
$$

where we have used the *a priori* assumption (4.1) in the last step of (4.24).

Lastly, we choose the positive constant δ_1 satisfying

$$C\sqrt{\delta_1}<\min\Big\{\frac{\mu_1}{2},K\Big\},$$

then the desired inequality (4.2) follows immediately. □

With the help of the standard boot-strap argument, for instance, see [15], Theorem 1.1 follows from the local existence result (Proposition 3.1) and *a priori* estimate (Proposition 4.1). Here, we give the outline of the proof.

Proof of Theorem 1.1. If the initial data satisfy $\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\leq\frac{\delta_1}{2}$, by Proposition 3.1, then we determine a time $T_1>0(T_1\leq T_0)$ such that the local solutions of (3.1)-(3.2) exists in $\widetilde{\mathcal{C}}_{T_1}(B^\sigma_{2,1})$ and $\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_{T_1}(B^\sigma_{2,1})}\leq\delta_1$. Therefore from Proposition 4.1 the solutions satisfy the *a priori* estimate $\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_{T_1}(B^\sigma_{2,1})}\leq C_1\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\leq\frac{\delta_1}{2}$ provided $\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\leq\frac{\delta_1}{2C_1}$. Thus by Proposition 3.1 the system (3.1)-(3.2) for $t\geq T_1$ with the initial data $(\varrho,\mathbf{v})(T_1)$ has again a unique solution $(\varrho,\mathbf{v})$ satisfying $\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_{(T_1,2T_1)}(B^\sigma_{2,1})}\leq\delta_1$, further $\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_{2T_1}(B^\sigma_{2,1})}\leq\delta_1$. Then by Proposition 4.1 we have $\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty_{2T_1}(B^\sigma_{2,1})}\leq C_1\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\leq\frac{\delta_1}{2}$. Thus we can continuous the same process for $0\leq t\leq nT_1,n=3,4,...$ and finally get a global solution $(\varrho,\mathbf{v})\in\widetilde{\mathcal{C}}(B^\sigma_{2,1})$ satisfying

$$
\begin{aligned}
&\|(\varrho,\mathbf{v})\|_{\widetilde{L}^\infty(B^\sigma_{2,1})}\\
&\quad+\lambda_1\Big\{\Big\|\frac{1}{\sqrt{\tau}}\mathbf{v}\Big\|_{\widetilde{L}^2(B^\sigma_{2,1})}+\Big\|\sqrt{\tau}\nabla\varrho\Big\|_{\widetilde{L}^2(B^{\sigma-1}_{2,1})}\Big\}\\
&\leq C_1\|(\varrho_0,\mathbf{v}_0)\|_{B^\sigma_{2,1}}\leq\frac{\delta_1}{2}.
\end{aligned}
\tag{4.25}
$$

The choice of δ_1 is sufficient to ensure $\frac{\gamma-1}{2}\varrho+\bar{\psi}>0$. Then it follows form Remark 3.1 that $(\rho,\mathbf{v})\in\mathcal{C}^1([0,\infty)\times\mathbb{R}^d)$ is a classical solution of (1.1)-(1.2) with $\rho>0$. Furthermore, we arrive at Theorem 1.1 with $\delta_0=\min(\delta_1/2,\delta_1/2C_1)$. □

5 Relaxation limit

In this section, we give the proof of Theorem 1.2.

Proof. From (1.7) and Remark 2.2, we deduce that quantities $\sup_{s\geq 0}\|\rho^\tau-\bar\rho\|_{B^\sigma_{2,1}}$ and

$$\frac{1}{\tau}\int_0^\infty \|\rho\mathbf{v}(t)\|^2_{B^\sigma_{2,1}}dt=\frac{1}{\tau^2}\int_0^\infty \|\rho^\tau\mathbf{v}^\tau(s)\|^2_{B^\sigma_{2,1}}ds$$

are bounded uniformly with respect to τ. Therefore, the left-hand side of (1.4) reads as $\tau^2\times$ the time derivative of a quantity which is bounded in $L^2(\mathbb{R}^+\times\mathbb{R}^d)$, plus $\tau^2\times$ the space derivative of a quantity which is bounded in $L^1(\mathbb{R}^+\times\mathbb{R}^d)$. So, this allows us to pass to the limit $\tau\to 0$ in the sense of distributions, and we arrive at

$$-\frac{\rho^\tau\mathbf{v}^\tau}{\tau}-\nabla P(\rho^\tau)\rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d).$$

Inserting the weak convergence property into the first equation of (1.4), we have

$$\partial_s\rho^\tau-\Delta P(\rho^\tau)\rightharpoonup 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d)$$

as $\tau\to 0$.

On the other hand, by (1.4), we conclude that $\partial_s\rho^\tau$ is bounded in $L^2(\mathbb{R}^+,B^{\sigma-1}_{2,1})$. Hence, it follows from Proposition 2.3 and Aubin-Lions compactness lemma in [18] that there exists some function $\mathcal{N}\in\mathcal{C}(\mathbb{R}^+,\bar\rho+B^\sigma_{2,1})$ such that as $\tau\to 0$, it holds that

$$\{\rho^\tau\}\to\mathcal{N} \quad \text{strongly in} \quad \mathcal{C}([0,T],(B^{\sigma-\delta}_{2,1})(B_r)),$$

for any $T>0$ and $\delta\in(0,1)$, which implies that $\mathcal{N}$ is a global weak solution to the porous medium equation (1.6) satisfying (1.8). For more details, the reader is referred to e.g. [3].

Therefore, the proof of Theorem 1.2 is complete. □

6 Appendix

As we known, Vishik, Bahouri, Chemin and Danchin *et al.* [20, 1, 5] have obtained some estimates of commutator, however, their results are unable to be applied to our case directly. Hence, following from their arguments, we develop a new estimate of commutator.

Proposition 6.1. *Let $s>0$, $1\leq r\leq\infty$ and $p,p_1,p_2\in[1,\infty]^3$ with $1/p=1/p_1+1/p_2$. There exists a generic constant $C>0$ depending only on p,p_1,p_2,σ,r,d such that*

$$2^{qs}\|[f,\Delta_q]\mathcal{A}g\|_{L^p}\leq Cc_q\|\nabla f\|_{L^{p_1}\cap B^{\sigma-1}_{p_2,r}\cap B^0_{p_1,r}}(\|g\|_{B^\sigma_{p_2,r}}+\|\nabla g\|_{L^{p_1}}), \tag{6.1}$$

where the operator $\mathcal{A}:=\mathrm{div}$ or ∇. As a direct consequence, when $1\leq p\leq p_2\leq p_1\leq\infty$, if

$$s>1+d\Big(\frac{1}{p_2}-\frac{1}{p_1}\Big) \quad \text{or} \quad s=1+d\Big(\frac{1}{p_2}-\frac{1}{p_1}\Big) \quad \text{and} \quad r=1,$$

then

$$2^{qs}\|[f,\Delta_q]\mathcal{A}g\|_{L^p}\leq Cc_q\|\nabla f\|_{B^{s-1}_{p_2,r}}\|g\|_{B^s_{p_2,r}}, \tag{6.2}$$

where $\{c_q\}$ denotes a sequence such that $\|(c_q)\|_{l^r}\leq 1$.

Proof. To show that the gradient part of f is involved in the estimate, we need to split f into low and high frequencies: $f = \Delta_{-1}f + \tilde{f}$. Obviously, there exists a constant $C > 0$ such that

$$\|\Delta_{-1}\nabla f\|_{L^p} \le C\|\nabla f\|_{L^p}, \quad \|\nabla\tilde{f}\|_{L^p} \le C\|\nabla f\|_{L^p}, \quad p \in [1,\infty]. \tag{6.3}$$

Since $\tilde{\varrho}$ is spectrally supported away from the origin, that is, there exists a radius $0 < R < \frac{3}{4}$ such that Supp $\mathcal{F}\tilde{f}\bigcap B(0,R) = \emptyset$, Lemma 2.1 implies

$$\|\Delta_q\nabla\tilde{f}\|_{L^p} \approx 2^q\|\Delta_q\tilde{f}\|_{L^p}, \quad p \in [1,\infty], \quad q \ge -1. \tag{6.4}$$

Without loss of generality, we proceed the proof with $\mathcal{A}g = \mathrm{div}g$. Taking advantage of Bony's decomposition, we have

$$\begin{aligned}
[f,\Delta_q]\mathrm{div}g &= [\tilde{f},\Delta_q]\mathrm{div}g + [\Delta_{-1}f,\Delta_q]\mathrm{div}g \\
&= \tilde{f}\Delta_q\mathrm{div}g - \Delta_q(\tilde{f}\mathrm{div}g) + [\Delta_{-1}f,\Delta_q]\mathrm{div}g \\
&= T_{\tilde{f}}\Delta_q\mathrm{div}g + T_{\Delta_q\mathrm{div}g}\tilde{f} + R(\tilde{f},\Delta_q\mathrm{div}g) \\
&\quad -\Delta_q(T_{\tilde{f}}\mathrm{div}g + T_{\mathrm{div}g}\tilde{f} + R(\tilde{f},\mathrm{div}g)) + [\Delta_{-1}f,\Delta_q]\mathrm{div}g.
\end{aligned}$$

Set $[f,\Delta_q]\mathrm{div}g \equiv \sum_{i=1}^{6} F_q^i$, where

$$\begin{aligned}
F_q^1 &= T_{\tilde{f}}\Delta_q\partial_j g^j - \Delta_q T_{\tilde{f}}\partial_j g^j, \qquad (\mathrm{div}g := \partial_j g^j) \\
F_q^2 &= T_{\Delta_q\partial_j g^j}\tilde{f}, \\
F_q^3 &= -\Delta_q T_{\partial_j g^j}\tilde{f}, \\
F_q^4 &= \partial_j R(\tilde{f},\Delta_q g^j) - \partial_j\Delta_q R(\tilde{f},g^j), \\
F_q^5 &= \Delta_q R(\partial_j\tilde{f},g^j) - R(\partial_j\tilde{f},\Delta_q g^j) \\
F_q^6 &= [\Delta_{-1}f,\Delta_q]\mathrm{div}g.
\end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned}
F_q^1 &= \sum_{q'} S_{q'-1}\tilde{f}\Delta_{q'}\Delta_q\partial_j g^j - \Delta_q\sum_{q'} S_{q'-1}\tilde{f}\Delta_{q'}\partial_j g^j \\
&= \sum_{|q-q'|\le 4} [S_{q'-1}\tilde{f},\Delta_q]\partial_j\Delta_{q'}g^j \\
&= \sum_{|q-q'|\le 4}\int_{\mathbb{R}^d} h(y)[S_{q'-1}\tilde{f}(x) - S_{q'-1}\tilde{f}(x-2^{-q}y)]\partial_j\Delta_{q'}g^j(x-2^{-q}y)dy.
\end{aligned}$$

Then, applying first order Taylor's formula, Young's inequality, Lemma 2.1 and (6.3), we get

$$\begin{aligned}
2^{q\sigma}\|F_q^1\|_{L^p} &\le C\sum_{|q-q'|\le 4}\|\nabla\tilde{f}\|_{L^{p_1}}2^{(\sigma-1)(q-q')}2^{q'\sigma}\|\Delta_{q'}g^j\|_{L^{p_2}} \\
&\le Cc_{q1}\|\nabla f\|_{L^{p_1}}\|g\|_{B^\sigma_{p_2,r}}, \qquad c_{q1} := \sum_{|q-q'|\le 4}\frac{2^{q'\sigma}\|\Delta_{q'}g\|_{L^{p_2}}}{9\|g\|_{B^\sigma_{p_2,r}}}.
\end{aligned}$$

and

$$
\begin{aligned}
2^{q\sigma}\|F_q^2\|_{L^p} &= 2^{q\sigma}\Big\|\sum_{q'\geq q-3} S_{q'-1}\partial_j\Delta_q g^j\Delta_{q'}\tilde{f}\Big\|_{L^p}\\
&\leq 2^{q\sigma}\sum_{q'\geq q-3}\|\Delta_{q'}\tilde{f}\|_{L^{p_1}}\|S_{q'-1}\partial_j\Delta_q g^j\|_{L^{p_2}}\\
&\leq C\sum_{q'\geq q-3}2^{q-q'}\|\nabla f\|_{L^{p_1}}2^{q\sigma}\|\Delta_q g\|_{L^{p_2}}\\
&\leq Cc_{q2}\|\nabla f\|_{L^{p_1}}\|g\|_{B^\sigma_{p_2,r}},\quad c_{q2}:=\frac{2^{q\sigma}\|\Delta_q g\|_{L^{p_2}}}{\|g\|_{B^\sigma_{p_2,r}}}.
\end{aligned}
$$

The third part F_q^3 is proceeded as follows:

$$
\begin{aligned}
F_q^3 &= -\Delta_q T_{\partial_j g^j}\tilde{f}\\
&= -\sum_{|q-q'|\leq 4}\Delta_q(S_{q'-1}\partial_j g^j\Delta_{q'}\tilde{f}),
\end{aligned}
$$

then

$$
\begin{aligned}
2^{q\sigma}\|F_q^3\|_{L^p} &\leq C\sum_{|q-q'|\leq 4}2^{(q-q')\sigma}2^{q'\sigma}\|S_{q'-1}\partial_j g^j\Delta_{q'}\tilde{f}\|_{L^p}\\
&\leq C\sum_{|q-q'|\leq 4}2^{(q-q')\sigma}\|S_{q'-1}\partial_j g^j\|_{L^{p_1}}2^{q'(\sigma-1)}\|\Delta_{q'}\nabla\tilde{f}\|_{L^{p_2}}\\
&\leq Cc_{q3}\|\nabla f\|_{B^{\sigma-1}_{p_2,r}}\|\nabla g\|_{L^{p_1}},\quad c_{q3}:=\sum_{|q-q'|\leq 4}\frac{2^{q'(\sigma-1)}\|\Delta_{q'}\nabla f\|_{L^{p_2}}}{9\|\nabla f\|_{B^{\sigma-1}_{p_2,r}}}.
\end{aligned}
$$

By the definition 2.2 and Proposition 2.1, we have

$$
\begin{aligned}
F_q^4 &= \partial_j R(\tilde{f},\Delta_q g^j)-\partial_j\Delta_q R(\tilde{f},g^j)\\
&= \sum_{|q-q'|\leq 1}\partial_j(\Delta_{q'}\tilde{f}\tilde{\Delta}_{q'}\Delta_q g^j)-\partial_j\Delta_q R(\tilde{f},g^j)\\
&= F_q^{4,1}+F_q^{4,2}.
\end{aligned}
$$

For the first term, using (6.4) and Lemma 2.1, we obtain

$$
\begin{aligned}
2^{q\sigma}\|F_q^{4,1}\|_{L^p} &\leq 2^{q\sigma}\sum_{|q-q'|\leq 1}\|\Delta_{q'}\nabla\tilde{f}\|_{L^{p_1}}\|\tilde{\Delta}_{q'}g^j\|_{L^{p_2}}+2^{q\sigma}\sum_{|q-q'|\leq 1}2^{q-q'}\|\Delta_{q'}\nabla\tilde{f}\|_{L^{p_1}}\|\tilde{\Delta}_{q'}g^j\|_{L^{p_2}}\\
&\leq C\|\nabla f\|_{L^{p_1}}\sum_{|q-q'|\leq 1}2^{(q-q')\sigma}2^{q'\sigma}\|\tilde{\Delta}_{q'}g^j\|_{L^{p_2}}\\
&\quad +C\|\nabla f\|_{L^{p_1}}\sum_{|q-q'|\leq 1}2^{(q-q')(\sigma+1)}2^{q'\sigma}\|\tilde{\Delta}_{q'}g^j\|_{L^{p_2}}\\
&\leq Cc_{q4(1)}\|\nabla f\|_{L^{p_1}}\|g\|_{B^\sigma_{p_2,r}},\quad c_{4(1)}:=\sum_{|q-q'|\leq 1}\frac{2^{q'\sigma}\|\Delta_{q'}g\|_{L^{p_2}}}{4\|g\|_{B^\sigma_{p_2,r}}}.
\end{aligned}
$$

The second term is estimated as:

$$\begin{aligned}
2^{q\sigma}\|F_q^{4,2}\|_{L^p} &= 2^{q\sigma}\|\partial_j\Delta_q R(\tilde f, g^j)\|_{L^p}\\
&\le C2^{q(\sigma+1)}\|\Delta_q R(\tilde f, g^j)\|_{L^p}\\
&\le Cc_{q4(2)}\|R(\tilde f, g^j)\|_{B^{\sigma+1}_{p,r}}\\
&\le Cc_{q4(2)}\|\tilde f\|_{B^1_{p_1,r_1}}\|g\|_{B^\sigma_{p_2,r_2}}\Big(\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}\Big)\\
&\le Cc_{q4(2)}\|\tilde f\|_{B^1_{p_1,r}}\|g\|_{B^\sigma_{p_2,r}}\\
&\le Cc_{q4(2)}\|\nabla\tilde f\|_{B^0_{p_1,r}}\|g\|_{B^\sigma_{p_2,r}}\\
&\le Cc_{q4(2)}\|\nabla f\|_{B^0_{p_1,r}}\|g\|_{B^\sigma_{p_2,r}}, \quad c_{q4(2)}:=\frac{2^{q(\sigma+1)}\|\Delta_q R(\tilde f, g^j)\|_{L^p}}{4\|R(\tilde f, g^j)\|_{B^{\sigma+1}_{p,r}}},
\end{aligned}$$

where we have used Lemma 2.2 and the result of continuity for the remainder (Proposition 2.2). Among them, $s+1>0$ is required.

For F_q^5, it follows from the same argument as F_q^4 that

$$2^{q\sigma}\|F_q^5\|_{L^p}\le Cc_{q5}\|\nabla f\|_{L^{p_1}\cap B^0_{p_1,r}}\|g\|_{B^\sigma_{p_2,r}},$$

where

$$c_{q5}:=\Big(\sum_{|q-q'|\le 1}\frac{2^{q'\sigma}\|\Delta_{q'}g\|_{L^{p_2}}}{4\|g\|_{B^\sigma_{p_2,r}}}\Big)+\frac{2^{q\sigma}\|\Delta_q R(\partial_j\tilde f, g^j)\|_{L^p}}{4\|R(\partial_j\tilde f, g^j)\|_{B^\sigma_{p,r}}},$$

and $s>0$ is required.

For $F_q^6=\sum_{|q-q'|\le 1}[\Delta_q(\Delta_{-1}f\partial_j\Delta_{q'}g^j)-\Delta_{-1}f\Delta_q\Delta_{q'}\partial_j g^j]$ ($g^j=\sum_{q'}\Delta_{q'}g^j$), by applying first order Taylor's formula, Young's inequality, Lemma 2.1 and (6.3), we have

$$\begin{aligned}
2^{q\sigma}\|F_q^6\|_{L^p} &= \Big\|\sum_{|q-q'|\le 1}\int_{\mathbb{R}^d}h(y)\Big[\Delta_{-1}f(x)+\Delta_{-1}f(x-2^{-q}y)\Big]\Delta_{q'}\partial_j g^j(x-2^{-q}y)dy\Big\|_{L^p}\\
&\le C\sum_{|q-q'|\le 1}2^{(q-q')(\sigma-1)}\|\nabla\Delta_{-1}f\|_{L^{p_1}}2^{q'\sigma}\|\Delta_{q'}g\|_{L^{p_2}}\\
&\le Cc_{q6}\|\nabla f\|_{L^{p_1}}\|g\|_{B^\sigma_{p_2,r}}, \quad c_{q6}:=\sum_{|q-q'|\le 1}\frac{2^{q'\sigma}\|\Delta_{q'}g\|_{L^{p_2}}}{3\|g\|_{B^\sigma_{p_2,r}}}.
\end{aligned}$$

Adding above these inequalities together, the inequality (6.1) is followed with $c_q=\frac{1}{6}\sum_{i=1}^6 c_{qi}$ satisfying $\|(c_q)\|_{\ell^r}\le 1$.

Furthermore, if

$$s>1+d\Big(\frac{1}{p_2}-\frac{1}{p_1}\Big)\quad\text{or}\quad s=1+d\Big(\frac{1}{p_2}-\frac{1}{p_1}\Big)\ \text{ and }\ r=1$$

with $1\le p\le p_2\le p_1\le\infty$, we have the following embedding properties:

$$B^{s-1}_{p_2,r}\hookrightarrow L^{p_1},\quad B^{s-1}_{p_2,r}\hookrightarrow B^{s-1-d(\frac{1}{p_2}-\frac{1}{p_1})}_{p_1,r}\hookrightarrow B^0_{p_1,r},$$

the inequality (6.2) follows immediately.

Therefore, the proof of Proposition 6.1 is complete. $\square$

Having Proposition 6.1, we may deal with some estimates of commutator of special form in the proof of *a priori* estimate, which are not covered by Lemma 2.3. For clarity, we give them by a corollary.

Corollary 6.1. *Let $\sigma = 1 + d/2$. There exists a generic constant $C > 0$ depending only on σ, d such that*

$$\begin{cases} 2^{q\sigma}\|[\varrho, \Delta_q]\mathrm{div}\mathbf{v}\|_{L^{2d/d+2}} \leq Cc_q\|\nabla\varrho\|_{B^{\sigma-1}_{2,1}}\|\mathbf{v}\|_{B^{\sigma}_{2,1}}; \\ 2^{q\sigma}\|[\mathbf{v}, \Delta_q]\cdot\nabla\varrho\|_{L^{2d/d+2}} \leq Cc_q\|\nabla\mathbf{v}\|_{B^{\sigma-1}_{2,1}}\|\varrho\|_{B^{\sigma}_{2,1}}; \end{cases} \tag{6.5}$$

where $\{c_q\}$ denotes a sequence such that $\|(c_q)\|_{l^1} \leq 1$.

Proof. In Proposition 6.1, it suffices to take

$$p = \frac{2d}{d+2}(d \geq 2),\ p_2 = 2,\ \sigma = 1 + \frac{d}{2}\ \text{ and }\ r = 1,$$

the conclusions follow obviously. □

According to Hölder inequality and Remark 2.2, it is not difficult to achieve the estimates of commutators in $L^r_T(L^{2d/d+2})$ spaces.

Corollary 6.2. *Let $\sigma = 1 + d/2$ and $1 \leq \theta \leq \infty$. Then there exists a generic constant $C > 0$ depending only on σ, d such that*

$$\begin{cases} 2^{q\sigma}\|[\varrho, \Delta_q]\mathrm{div}\mathbf{v}\|_{L^{\theta}_T(L^{2d/d+2})} \leq Cc_q\|\nabla\varrho\|_{\widetilde{L}^{\theta_1}_T(B^{\sigma-1}_{2,1})}\|\mathbf{v}\|_{\widetilde{L}^{\theta_2}_T(B^{\sigma}_{2,1})}; \\ 2^{q\sigma}\|[\mathbf{v}, \Delta_q]\cdot\nabla\varrho\|_{L^{\theta}_T(L^{2d/d+2})} \leq Cc_q\|\nabla\mathbf{v}\|_{\widetilde{L}^{\theta_1}_T(B^{\sigma-1}_{2,1})}\|\varrho\|_{\widetilde{L}^{\theta_2}_T(B^{\sigma}_{2,1})}; \end{cases} \tag{6.6}$$

where $\{c_q\}$ denotes a sequence such that $\|(c_q)\|_{l^1} \leq 1$ and $\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$.

Remark 6.1. Actually, if we take $p = p_2,\ p_1 = \infty,\ \sigma = 1 + d/p$ and $r = 1$ in Proposition 6.1, we can also deduce Lemma 2.3.

Acknowledgments

The research of Jiang Xu is partially supported by the NSFC (11001127), China Postdoctoral Science Foundation (20110490134) and NUAA Research Funding (NS2010204). The research of Zejun Wang is partially supported by the NSFC(10901082) and China Postdoctoral Science Foundation (20090450149).

References

[1] H. Bahouri, J. Y. Chemin and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, Berlin, Heidelberg: Springer-Verlag, 2011.

[2] J. Y. Chemin, Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel, *Journal d'Analyse Mathématique* **77** (1999) 25–50.

[3] J.-F. Coulombel and T. Goudon, The strong relaxation limit of the multidimensional isothermal Euler equations, *Trans. Amer. Math. Soc.* **359** (2007) 637–648.

[4] C. M. Dafermos. Can dissipation prevent the breaking of waves? In: Transactions of the Twenty-Sixth Conference of Army Mathematicians, 187C198, ARO Rep. 81, 1, U. S. Army Res. Office, Research Triangle Park, N.C., 1981.

[5] R. Danchin, *Fourier Analysis Methods for PDE's* (Lecture Notes), 2005.

[6] L. C. Evans, *Partial differential equations*, Providence, Rhode Island: Amer Mathematical Society, 1998.

[7] D. Y. Fang and J. Xu, Existence and asymptotic behavior of $\mathcal{C}^1$ solutions to the multidimensional compressible Euler equations with damping, *Nonlinear Anal. TMA* **70** (2009) 244–261.

[8] L. Hsiao. *Quasilinear Hyperbolic Systems and Dissipative Mechanisms*, Singapore: World Scientific Publishing, 1997.

[9] F. Huang and R. Pan, Convergence rate for compressible Euler equations with damping and vacuum, *Arch. Rational Mech. Anal.* **166** (2003) 359–376.

[10] F. Huang, P. Marcati and R. Pan, Convergence to Barenblatt solution for the compressible Euler equations with damping and vacuum, *Arch. Rational Mech. Anal.* **176** (2005) 1–24.

[11] S. Junca and M. Rascle, Strong relaxation of the isothermal Euler system to the heat equation, *Z. Angew. Math. Phys.* **53** (2002) 239–264.

[12] A. Majda, *Compressible Fluid Flow and Conservation Laws in Several Space Variables*, Berlin, New York: Springer-Verlag, 1984.

[13] P. Marcati and A. Milani, The one-dimensional Darcy's law as the limit of a compressible Euler flow, *J. Differential Equations* **84** (1990) 129–147.

[14] P. Marcati, A. Milani and P. Secchi, Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system, *Manuscripta Math.*, **60** (1988) 49-69.

[15] A. Matsumura and T. Nishida, The initial value problem for the quations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* **20** (1980) 67–104.

[16] P. Marcati and B. Rubino, Hyperbolic to parabolic relaxation theory for quasilinear first order systems, *J. Differential Equations* **162** (2000) 359–399.

[17] T. Nishida, Nonlinear hyperbolic equations and relates topics in fluid dynamics, *Publ. Math. D'Orsay* (1978) 46–53.

[18] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Math. Pura Appl.*, **146** (1987) 65–96.

[19] T. Sideris, B. Thomases and D. H. Wang, Long time behavior of solutions to the 3D compressible Euler with damping, *Comm. P. D. E.* **28** (2003) 953–978.

[20] M. Vishik, Hydrodynamics in Besov spaces, *Arch. Rational Mech. Anal.* **145** (1998) 197–214.

[21] W. Wang and T. Yang, The pointwise estimates of solutions for Euler equations with damping in multi-dimensions. *J Differential Equations* **173** (2001) 410–450.

[22] J. Xu, Global classical solutions to the compressible Euler-Maxwell equations. *SIAM J. Math. Anal.*, 2011, in press.